Limit theorems for geometrically ergodic Markov chains

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LIMIT THEOREMS FOR
GEOMETRICALLY ERGODIC MARKOV CHAINS

Abbreviated title : Limit theorems for geometrically ergodic Markov chains

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Abstract. *Let* $(E, \mathcal{E})$ *be a countably generated state space, let* $(X_n)_n$ *be an aperiodic and* ψ*-irreducible* V*-geometrically ergodic Markov chain on* E*, where* V *is a function from* E *to* $[1, +\infty[$ *and* ψ *is a* σ*-finite positive measure on* E*. Let* π *be the* P*-invariant distribution, and let* ξ *be a real-valued measurable function on* E *which is supposed to be dominated by* $\sqrt{V}$*. We know that* $\sigma^2 = \lim_n n^{-1} \mathbb{E}_x[(S_n)^2]$ *exists for any* $x \in E$ *(and does not depend on* x*), and that, in the case* $\sigma^2 > 0$*,* $n^{-\frac{1}{2}}[\xi(X_1) + \ldots + \xi(X_n) - n\pi(\xi)]$ *converges in distribution to the normal distribution* $\mathcal{N}(0, \sigma^2)$*.*
In this work we prove that, for any initial distribution μ_0 *satisfying* $\mu_0(V) < +\infty$ *and under the condition* $\sigma^2 > 0$*,*
- If ξ *is dominated by* V^α *with* $\alpha < \frac{1}{4}$*, then the rate of convergence in the central limit theorem is* $O(\frac{1}{\sqrt{n}})$*.*
- If ξ *is dominated by* V^α *with* $\alpha < \frac{1}{2}$*, then* $(\xi(X_n))_n$ *satisfies a local limit theorem under a usual non-arithmeticity assumption.*

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I. STATEMENTS OF RESULTS

Let $(E, \mathcal{E})$ be a measurable space with countably generated σ-field $\mathcal{E}$, and let $(X_n)_{n\geq 0}$ be a Markov chain on E with transition probability, P. We assume that

Hypothesis. *$(X_n)_{n\geq 0}$ is aperiodic, ψ-irreducible for a certain positive σ-finite measure ψ on E, and there exists a function V defined on E, taking values in $[1, +\infty[$, such that $(X_n)_{n\geq 0}$ is V-geometrically ergodic, that is :*

- *there exists a P-invariant probability measure, π, satisfying $\pi(V) < +\infty$*
- *there exist real numbers $\kappa < 1$ and $C \geq 0$ such that we have, for all $n \geq 1$ and $x \in E$,*

$$(*) \qquad \sup\Big\{ |P^n f(x) - \pi(f)| \, , \ f : E \to \mathbb{C} \text{ measurable}, \ |f| \leq V \Big\} \leq C\, \kappa^n \, V(x).$$

Recall that ψ-irreducibility means that, for all $x \in E$ and $A \in \mathcal{E}$ such that $\psi(A) > 0$, there exists $n = n(x, A) \in \mathbb{N}^*$ such that $P^n 1_A(x) > 0$. Many examples of geometrically ergodic Markov chains can be found in Meyn and Tweedie (1993).

Let ξ be a real-valued π-integrable function on E such that $\pi(\xi) = 0$. If not, Theorems I-II will apply to the function $\xi - \pi(\xi)$. For $n \geq 1$, we set

$$S_n = \sum_{k=1}^{n} \xi(X_k).$$

If ξ is dominated by $\sqrt{V}$, we know that the sequence $(\frac{1}{n}\mathbb{E}_x[(S_n)^2])_n$ converges to a non-negative real number σ^2 (the asymptotic variance) whose value does not depend on the initial condition $X_0 = x$, $x \in E$, and that, in the case $\sigma^2 > 0$, the sequence of r.v $(\frac{S_n}{\sqrt{n}})_n$ converges in distribution to the normal distribution $\mathcal{N}(0, \sigma^2)$. See for instance Meyn and Tweedie (1993) Chap. 17, Chen, X. (1999) (case $V \equiv 1$), and Kontoyiannis and Meyn (2003) for the caracterization of the case $\sigma^2 = 0$.

Let $0 \leq \alpha < \frac{1}{2}$. In this paper, we shall suppose that ξ is dominated by V^α, that is $\dfrac{\xi}{V^\alpha}$ is bounded on E, and we intend to establish a CLT with rate of convergence and a local limit theorem under simple additional assumptions on α that will be specified in each of these statements. [1]

The initial distribution of the chain is denoted by μ_0. The condition $\mu_0(V) < +\infty$ used in Theorems I-II holds for instance when $\mu_0 = \pi$ (stationary case) and $\mu_0 = \delta_x$ for any $x \in E$ $(X_0 = x)$.

Theorem I (T.L.C with rate of convergence). [2] *If ξ is dominated by V^α, with $\alpha < \frac{1}{4}$, if $\sigma^2 > 0$ and $\mu_0(V) < +\infty$, then there exists a positive constant C such that we have for all $n \geq 1$*

$$\sup_{u \in \mathbb{R}} \left| \mathbb{P}_{\mu_0}\left[S_n \leq u\sigma\sqrt{n} \right] - \mathcal{N}(0,1)(]-\infty, u]) \right| \leq C \, \frac{1 + \mu_0(V)}{\sqrt{n}}.$$

[1] The first version of this work contained a renewal theorem. This theorem has been removed in this new version. Indeed, since a V-geometrically ergodic chain is Harris recurrent, the renewal theorem follows from a general work by Alsmeyer, published in *Markov Proc. Rel. Fields* **3**, *103-127.*

[2] Theorem I actually holds with $\alpha = \frac{1}{3}$, See the recent work Hervé, L. (2006).

Recall that $A \in \mathcal{E}$ is said to be P-invariant if $P(a, A) = 1$ for all $a \in A$.
In local theorem (Th. II), we shall need the following non-arithmeticity assumption whose functional meaning will be investigated in Section III.3. Let $0 < \theta \leq 1$.

Hypothesis (N-A)$_\theta$. There is no $t \in I\!R^*$, no P-invariant subset $A \in \mathcal{E}$, no $\lambda \in \mathbb{C}$, $|\lambda| = 1$, and no functions w dominated by V^θ, with non-zero constant modulus on A, such that we have

$$\forall x_0 \in A,\ \forall n \geq 1,\ exp\Big(it[\xi(x_1) + \ldots + \xi(x_n)]\Big)\, w(x_n) = \lambda^n w(x_0)\ \ \Pi_{i=1}^n P(x_{i-1}, dx_i) - p.s.$$

Theorem II (local). [3] *If (N-A)$_\theta$ holds for a certain $\theta \in]0,1]$, if ξ is dominated by V^α, with $\alpha < \frac{1}{2}$, if $\sigma^2 > 0$ and $\mu_0(V) < +\infty$, then we have for every finite interval $J = [a, b]$ of $I\!R$*

$$\lim_n \sigma\sqrt{2\pi n}\, I\!P_{\mu_0}([S_n \in J]) = b - a.$$

In the stationary case, Theorem I is provided by the theorem of Bolthausen, E. (1982) under the weaker condition $\pi(|\xi|^3) < +\infty$. In Kontoyiannis and Meyn (2003), Theorem I is proved in the non-stationary case for bounded functions ξ. In Steinsaltz, D. (2001), as in the present work, ξ is supposed to be dominated by V^α, but the rate of convergence obtained by Steinsaltz is $O((\frac{\ln n}{n})^\beta)$, with $\beta = \frac{1}{2(\alpha+1)}$ and $\alpha \leq \frac{1}{2}$. In Fuh, C.D.(1999) the rates of convergence in the CLT, which are expressed in terms of Paley's inequalities, are proved in the stationary case and under the conditon that $\frac{P(\xi^2 V)}{V}$ is bounded, but in most cases this last condition requires moment conditions of exponential type on the transition probability P, See § II.2. Finally, to our knowledge, when ξ is not bounded, local theorem has not been investigated in the case of geometrically ergodic chains.

The proofs of Theorems I-II are based on the spectral method of Nagaev, initiated in Nagaev, S.V. (1957), and Guivarc'h and Hardy (1988), which consists in applying a perturbation theorem to the Fourier operators $P(t)f = P(e^{it\xi} f)$. This method, fully described in Hennion and Hervé (2001), requires a quasi-compact action of the transition probability P on a certain Banach space $\mathcal{B}$ composed of measurable functions on E. Condition $(*)$ ensures here that $\mathcal{B}$ may be the space $\mathcal{B}_V$ of measurable functions on E that are dominated by V, equipped with the norm $\|f\|_V = \sup_{x\in E} \frac{|f(x)|}{V(x)}$. Notice that spectral methods in $\mathcal{B}_V$-type spaces have been already applied in the context of geometrically ergodic chains, in particular in the above cited papers Fuh, C.D.(1999), Kontoyiannis and Meyn (2003). See also Meyn and Tweedie (1994), Kontoyiannis and Meyn (2005).

If ξ is bounded, then it is easily shown that, for $z \in \mathbb{C}$, the (Laplace) kernels $P_z(x, dy) = e^{z\xi} P(x, dy)$ define continuous actions P_z on the space $\mathcal{B}_V$ and that the map $z \mapsto P_z$ is analytic (use for instance the proof of Lemma VIII.10 in Hennion and Hervé (2001)). Consequently, if ξ is bounded, the standard perturbation theory applies to P_z and the general statements in Hennion and Hervé (2001) provide Theorems I-II and the large deviations theorem under a usual non-arithmeticity condition. However let us remember that the hypothesis that ξ is bounded is often too restrictive in applications.
If ξ is unbounded, there is no guarantee that the fonction $t \mapsto P(t)$ is continuous from $I\!R$ to the space of bounded operators on $\mathcal{B}_V$, See Section II.2, so one cannot employ the standard perturbation theory.

[3] Theorem II actually holds with $\alpha = \frac{1}{2}$. Indeed, from Section II.3, one can apply the results of Hervé, L. (2005).

As already exploited in Hennion and Hervé (2004) and Hervé, L. (2005), the use of the perturbation theorem of Keller and Liverani (1999) greatly enhances the power of the spectral method. Here it will enable us to relax the boundedness hypothesis on the functional ξ. We shall follow a pattern similar to that developped in Hennion and Hervé (2004) and in Hervé, L. (2005) :
- In Section II.3, we shall apply to $P(t)$ the theorem of Keller and Liverani ; this will be possible with the help of a recent work of Hennion, H. concerning quasi-compactness of operators acting on spaces of bounded functions.
- In Sections II.4, we shall prove Taylor's expansions of $P(t)$ at $t = 0$; in this part $P(t)$ will be seen as a bounded linear map between two suitable $\mathcal{B}_V$-type spaces.
- Then, in Section III, we shall be in a position to apply the techniques of characteristic functions described in Hennion and Hervé (2001) which are similar to those used for sums of independant identically distributed random variables.

Finally let us mention that, by using the method developped in Hennion and Hervé (2001) (2004), Theorems I-II can be extended to the sequence of r.v $(X_n, S_n)_n$.

This work has not been published in this form. Indeed, more precise results concerning the local theorem and the rate of convergence have been recently obtained for $\mathcal{B}$-geometrically ergodic chains, See Hervé, L. (2005) (2006).

II. STUDY OF FOURIER KERNELS

II.1 NOTATIONS.

If U is a function defined on E and taking values in $[1, +\infty[$, we denote by $(\mathcal{B}_U, \|\cdot\|_U)$ the Banach space of measurable complex-valued functions f on E such that

$$\|f\|_U = \sup_{x\in E} \frac{|f(x)|}{U(x)} < +\infty.$$

We denote by $\mathcal{L}(\mathcal{B}_U)$ the space of bounded operators on $\mathcal{B}_U$, by $\mathcal{B}'_U$ the topological dual space of $\mathcal{B}_U$. If $\phi \in \mathcal{B}'_U$ and $f \in \mathcal{B}_U$, we set $\phi(f) = \langle \phi, f\rangle$. For simplicity, $\|\cdot\|_U$ equally stands for the norm on $\mathcal{B}'_U$ and the operator norm on $\mathcal{L}(\mathcal{B}_U)$. Finally we set $\mathbf{1} = 1_E$.

Observe that a probability measure μ on E defines an element of $\mathcal{B}'_U$ if $\mu(U) < +\infty$; in particular we have $\pi \in \mathcal{B}'_V$ by hypothesis. Besides, the fact that f is dominated by V^α is equivalent to $f \in \mathcal{B}_{V^\alpha}$. Finally Hypothesis $(*)$ can be rewritten as

$$(**) \qquad \forall n \geq 1, \;\; \forall f \in \mathcal{B}_V, \qquad \|P^n f - \pi(f)\,\mathbf{1}\|_V \leq C\,\kappa^n\,\|f\|_V.$$

Equivalently this means that P is a power-bounded quasi-compact operator on $\mathcal{B}_V$ and that 1 is a simple eigenvalue and the unique eigenvalue of modulus one.

For $0 < \theta \leq 1$, we set $\mathcal{B}_\theta = \mathcal{B}_{V^\theta}$ and $\|\cdot\|_\theta = \|\cdot\|_{V^\theta}$, in particular $\mathcal{B}_1 = \mathcal{B}_V$ and $\|\cdot\|_1 = \|\cdot\|_V$.

Lemma 1. *For all $0 < \theta \leq 1$, $(X_n)_{n\geq 0}$ is V^θ-geometrically ergodic, that is : there exist real numbers $\kappa_\theta < 1$ and $C_\theta \geq 0$ such that*

$$(***) \qquad \forall n \geq 1, \;\; \forall f \in \mathcal{B}_\theta, \qquad \|P^n f - \pi(f)\,\mathbf{1}\|_\theta \leq C_\theta\,\kappa_\theta^n\,\|f\|_\theta.$$

Proof. This is a well-known result whose we briefly recall the proof. Under the aperiodicity and ψ-irreducibility hypotheses, condition $(*)$ is equivalent to the so-called drift criterion, See Meyn and Tweedie (1993) : there exist $r < 1$, $M \geq 0$, a petite set $C \in \mathcal{E}$, and a positive function W on E, satisfying $c^{-1}V \leq W \leq cV$ with $c \in \mathbb{R}_+^*$, such that $PW \leq rW + M1_C$. Since $u \mapsto u^\theta$ is concave, we get by Jensen's inequality $P(W^\theta) \leq (rW + M1_C)^\theta \leq r^\theta W^\theta + M^\theta 1_C$. It follows that $(X_n)_{n\geq 0}$ is V^θ-geometrically ergodic. □

II.2. FOURIER KERNELS AND PRELIMINARY REMARKS

The starting point in the spectral method is given by the following formula

$$(F) \qquad \forall t \in \mathbb{R}, \quad \mathbb{E}_{\mu_0}[e^{itS_n}] = \mu_0(P(t)^n \mathbf{1}) \qquad \text{(cf. Prop. 3, Sect. III.1)},$$

where μ_0 is the initial distribution and the $P(t)$'s denote the Fourier kernels associated to P and ξ, defined by

$$t \in \mathbb{R},\ x \in E, \qquad P(t)f(x) = P(e^{it\xi}f)(x) = \int_E e^{it\xi(y)} f(y)P(x, dy),$$

where f is any $P(x,\cdot)$-integrable function.
The spectral method requires a perturbation theorem that enables to generalize to $P(t)$ the properties $(**)$ or $(***)$ of Section II.1. Unfortunately, when ξ is unbounded, the standard perturbation theory does not apply in general. Actually, establishing even the continuity of the function $P(\cdot)$, taking values in $\mathcal{L}(\mathcal{B}_V)$, seems to be difficult. Indeed observe that we have for $f \in \mathcal{B}_V$

$$(C) \quad |P(t)f - Pf| \ \leq\ P(|e^{it\xi} - 1|\,|f|) \ \leq\ \|f\|_V P(|e^{it\xi} - 1|V) \ \leq\ |t|\,\|f\|_V P(|\xi|V),$$

hence, under the condition $|\xi| \leq D V^\alpha$, with $D \in \mathbb{R}_+$,

$$\|P(t) - P\|_V \ \leq\ |t| \sup_{x\in E} \frac{P(|\xi|V)(x)}{V(x)} \leq D\,|t| \sup_{x\in E} \frac{P(V^{1+\alpha})(x)}{V(x)}.$$

If $\alpha > 0$, then in general the function $\frac{PV^{1+\alpha}}{V}$ is not bounded on E. Let us observe that this problem always occurs in the specral method when ξ is unbounded, See for instance Milhaud and Raugi (1989), Hennion and Hervé (2001) (2004), Hervé, L. (2005).

As mentionned in Section I, it is supposed in Fuh, C.D.(1999) that $\frac{P(\xi^2V)}{V}$ is bounded. Notice that, under this condition, one can apply to $P(t)$ the standard perturbation theory (use the above inequalities). However this hypothesis is quite restrictive, actually it seems that it only holds for exponential-type functions V provided that the transition probability satisfies some moment conditions of exponential type.

II.3. PERTURBATION THEOREM OF KELLER-LIVERANI

As in Hennion and Hervé (2004) and in Hervé, L. (2005), the perturbation theorem of Keller and Liverani (1999) can be used in this work as a substitute for the standard perturbation theory. This theorem ensures the following spectral properties for which it is only assumed that ξ is measurable on E :

Proposition 1. *Let $0 < \theta \leq 1$. There exist an open interval I_θ containing $t = 0$, some real numbers $C_\theta \in I\!R_+$ and $\rho_\theta < 1$, and lastly some functions $\lambda(\cdot)$, $v(\cdot)$, $\phi(\cdot)$, and $N(\cdot)$, defined on I_θ and taking values in respectively $\mathbb{C}$, $\mathcal{B}_\theta$, $\mathcal{B}'_\theta$ and $\mathcal{L}(\mathcal{B}_\theta)$, such that we have, for all $n \geq 1$, $t \in I_\theta$, and $f \in \mathcal{B}_\theta$*

$$P(t)^n f = \lambda(t)^n \langle \phi(t), f \rangle \, v(t) \; + \; N(t)^n f,$$

with $\lim_{t\to 0} \lambda(t) = \lambda(0) = 1, \quad \lim_{t\to 0} v(t) = v(0) = \mathbf{1}, \quad \lim_{t\to 0} \phi(t) = \phi(0) = \pi, \quad \|N(t)^n\|_\theta \leq C_\theta \rho_\theta^n,$

$$\langle \phi(t), v(t) \rangle = 1, \quad N(t)v(t) = 0, \quad \phi(t)N(t) = 0, \quad \langle \pi, v(t) \rangle = 1.$$

Furthermore let ρ'_θ be any real number such that $\rho_\theta < \rho'_\theta < 1$, and let $\mathcal{R}_\theta$ be the subset of the complex plane defined by

$$\mathcal{R}_\theta = \{z : z \in \mathbb{C}, |z| \geq \rho_\theta, \; |z - 1| \geq 1 - \rho'_\theta\}.$$

Then

$$M_\theta = \sup \left\{ \|(z - P(t))^{-1}\|_\theta, \; t \in I_\theta, \; z \in \mathcal{R}_\theta \right\} < +\infty.$$

Since $P(t)$ is seen as an operator on $\mathcal{B}_\theta$, the above eigen-element of $P(t)$ should be denoted by λ_θ, v_θ, ϕ_θ, and N_θ. But, since for $\theta' \leq \theta$ the canonical embedding from $\mathcal{B}_{\theta'}$ into $\mathcal{B}_\theta$ is continuous, it is easily seen that these elements do not depend on θ.

The results in Keller and Liverani (1999) require the notion of essential spectral radius whose we recall the definition. Let $\mathcal{B}$ be a Banach space, let $T \in \mathcal{L}(\mathcal{B})$, and let $r(T)$ be the spectral radius of T. We denote by $T_{|G}$ the restriction of T to any T-invariant subspace G of $\mathcal{B}$.
The essential spectral radius of T, denoted by $r_e(T)$, is the greatest lower bound of $r(T)$ and the real numbers $r \geq 0$ for which there exists a decomposition into closed T-invariant subspaces

$$\mathcal{B} = F_r \oplus H_r,$$

where F_r has finite dimension, each eigenvalue of $T_{|F_r}$ is of modulus $\geq r$, while $r(T_{|H_r}) < r$.
In particular T is quasi-compact if and only if $r_e(T) < r(T)$.

Proposition 1 derives from the theorem of Keller and Liverani (1999) which can be applied with the help of the following statement. We conserve the notations of Lemma 1.

Proposition 2. *Let $0 < \theta \leq 1$. For all $t \in I\!R$, $P(t)$ is a bounded operator on $\mathcal{B}_\theta$, and the conditions of Keller-Liverani theorem hold, that is :*

(KL1) $\forall t \in I\!R, \quad \forall n \geq 1, \quad \forall f \in \mathcal{B}_\theta, \quad \|P(t)^n f\|_\theta \leq C_\theta \, \kappa_\theta^n \, \|f\|_\theta + \|1\|_\theta \, \pi(|f|).$

(KL2) There exists $\vartheta_\theta < 1$ such that, for all $t \in I\!R$, the essential spectral radius of $P(t)$ acting on $\mathcal{B}_\theta$ satisfies $r_e(P(t)) \leq \vartheta_\theta$.

(KL3) $\forall t \in I\!R, \; \forall f \in \mathcal{B}_\theta, \; \forall n \geq 1, \quad \pi(|P(t)^n f|) \leq \pi(|f|)$

(KL4) $\lim_{t\to 0} \; \sup \left\{ \pi\left(|P(t)f - Pf|\right), \; f \in \mathcal{B}_\theta, \; \|f\|_\theta \leq 1 \right\} \; = \; 0.$

Proof of (KL1), (KL3) and (KL4). It is easily proved by induction that $|P(t)^n f| \le P^n|f|$ for all $f \in \mathcal{B}_\theta$ and $n \ge 1$. Since $P \in \mathcal{L}(\mathcal{B}_\theta)$ (Lemma 1), this inequality applied with $n = 1$ implies that $P(t) \in \mathcal{L}(\mathcal{B}_\theta)$. By using the inequality $(***)$ of Lemma 1, and the invariance of π, the above inequality easily provides (KL1), and (KL3). Moreover observe that

$$\pi(|P(t)f - Pf|) \le \pi(P(|e^{it\xi} - 1|\,|f|)) = \pi(|e^{it\xi} - 1|\,|f|) \le \|f\|_\theta\, \pi(|e^{it\xi} - 1|\, V).$$

Then (KL4) follows from hypothesis $\pi(V) < +\infty$ and from Lebesgue's theorem. □

In order to establish (KL2), we shall use a recent result of Hennion, H. Denote by $(\tilde{\mathcal{B}}, \|\cdot\|_0)$ the space of bounded measurable complex-valued functions on E, equipped with its usual norm $\|g\|_0 = \sup_{x\in E} |g(x)|$.

Let Q be a bounded positive kernel on E, and let ν be a probability measure on E. We assume that there exist $\eta > 0$, $\vartheta < 1$ and $\ell \ge 1$ such that

$$(\mathcal{D}) \qquad \forall A \in \mathcal{E}, \quad \Big[\nu(A) \le \eta\Big] \ \Rightarrow\ \Big[\forall x \in E,\ \ Q^\ell(x, A) \le \vartheta^\ell\Big].$$

If Q is markovian, then this hypothesis corresponds to the well-known Doeblin condition.
Now let χ be a bounded measurable complex-valued function on $E \times E$, and let Q_χ be the bounded operator on $\tilde{\mathcal{B}}$ defined by $(Q_\chi g)(x) = \int_E g(y)\, \chi(x, y)\, Q(x, dy)$.

Theorem. [See Lemma III.4 in Hennion H.]
Under Condition $(\mathcal{D})$, we have $r_e(Q_\chi) \le \vartheta \sup_{x,y\in E} |\chi(x, y)|$.

Proof of (KL2). For $T \in \mathcal{L}(\mathcal{B}_\theta)$ and $g \in \tilde{\mathcal{B}}$, we set $\tilde{T}g = V^{-\theta}\, T(V^\theta\, g)$. Then

Lemma 2. *We have $r_e(T) = r_e(\tilde{T})$.*

Proof. Let $r \ge 0$. Assume that F_r and H_r are closed T-invariant subspaces of $\mathcal{B}_\theta$ such that

$$\mathcal{B}_\theta = F_r \oplus H_r,$$

where $\dim F_r < +\infty$, each eigenvalue of $T_{|F_r}$ is of modulus $\ge r$, and $r(T_{|F_r}) < r$. Then

$$\tilde{F}_r = \{g = V^{-\theta} f,\ f \in F_r\} \quad \text{and} \quad \tilde{H}_r = \{g = V^{-\theta} f,\ f \in H_r\}$$

are clearly closed $\tilde{T}$-invariant subspaces of $\tilde{\mathcal{B}}$. We have $\dim \tilde{F}_r = \dim F_r < +\infty$, and $\tilde{\mathcal{B}} = \tilde{F}_r \oplus \tilde{H}_r$. Moreover, if $g \in \tilde{F}_r$ is an eigen-function for $\tilde{T}$, then so is the function $V^\theta g \in F_r$ with respect to T and the same eigenvalue. Thus $\tilde{T}_{|\tilde{F}_r}$ has only eigenvalues of modulus $\ge r$. Lastly, from the definition of $\tilde{H}_r$ and $\|\cdot\|_\theta$, we get $\|(\tilde{T}_{|\tilde{H}_r})^n\|_0 = \|(T_{|H_r})^n\|_\theta$, thus $r(\tilde{T}_{|\tilde{H}_r}) < r$. Hence $r_e(\tilde{T}) \le r_e(T)$. On the same way we have $r_e(T) \le r_e(\tilde{T})$. □

Lemma 3. *The positive kernel $\widetilde{P}(x, dy) = V^{-\theta}(x)\, V^\theta(y)\, P(x, dy)$ satisfies the condition $(\mathcal{D})$ with respect to the probability measure ν_θ defined by $\nu_\theta(A) = \dfrac{1}{\pi(V^\theta)}\, \pi(V^\theta\, 1_A)$ for all $A \in \mathcal{E}$.*

Proof. Observe that $\widetilde{P^n} = (\tilde{P})^n$. Then Lemma 3 is an easy consequence of the inequality $(***)$ of Lemma 1 and the fact that $V^\theta \geq 1$. $\square$

Now let $t \in I\!R$, and apply the above theorem with $Q = \widetilde{P}$ and $\chi(x, y) = e^{it\xi(y)}$ (the variable x does not occur here). We have $Q_\chi = \widetilde{P(t)}$ and $|\chi| = 1$, thus $r_e(\widetilde{P(t)}) \leq \vartheta_\theta$, where ϑ_θ is a real number in $]0, 1[$ which is given by Lemma 3 (notice that ϑ_θ does not depend on t). Then, by Lemma 2, we get $r_e(P(t)) \leq \vartheta_\theta$. $\square$

II.4. TAYLOR'S EXPANSIONS OF $P(t)$ AND OF ITS EIGEN-ELEMENTS

In order to establish limit theorems of Section I, we have to obtain some expansions at $t = 0$ of the functions $\lambda(\cdot)$, $v(\cdot)$, $\phi(\cdot)$ and $N(\cdot)$ introduced in Proposition 1. We proceed as in Hennion and Hervé (2004).

Let $\|\cdot\|_{\theta',\theta}$ be the operator norm in the space $\mathcal{L}(\mathcal{B}_{\theta'}, \mathcal{B}_\theta)$ of bounded linear maps from $\mathcal{B}_{\theta'}$ to $\mathcal{B}_\theta$. We assume that ξ is dominated by V^α, with $0 < \alpha \leq \frac{1}{2}$, that is there exists $D \in I\!R_+$ such that we have for all $x \in E$

$$|\xi(x)| \leq D\, V^\alpha(x).$$

For $0 < \theta' < \theta$ and $n \in I\!N$, we introduce the following condition on α

$$\mathcal{U}_n(\theta', \theta) \ : \quad \theta \in\,]\theta' + n\alpha, 1],$$

and we shall write TE(n) for Taylor's expansion of order n. In the next Lemma, $\rho_{\theta'}$ denotes the real number in $]0, 1[$ defined in Proposition 1 (applied to $P(t)$ acting on $\mathcal{B}_{\theta'}$).

Lemma 4.
(a) For $n \in I\!N$, under condition $\mathcal{U}_n(\theta', \theta)$, $P(t)$ has a TE(n) at $t = 0$ in $\mathcal{L}(\mathcal{B}_{\theta'}, \mathcal{B}_\theta)$.

(b) Under condition $\mathcal{U}_1(\theta', \theta)$, there exist some real numbers a, b satisfying $\rho_{\theta'} < a < b < 1$, and a continuous function $z \mapsto R'_z$ from $\mathcal{R} = \{z : z \in \mathbb{C}\ , |z| \geq a,\ |z - 1| \geq 1 - b\}$ to $\mathcal{L}(\mathcal{B}_{\theta'}, \mathcal{B}_\theta)$ such that

$$\lim_{t\to 0} \frac{1}{|t|} \sup_{z\in\mathcal{R}} \left\| (z - P(t))^{-1} - (z - P)^{-1} - tR'_z \right\|_{\theta',\theta} = 0.$$

(c) Under condition $\mathcal{U}_2(\theta', \theta)$, there exist some real numbers a, b satisfying $\rho_{\theta'} < a < b < 1$, and some continuous functions $z \mapsto R'_z$ and $z \mapsto R''_z$ from $\mathcal{R} = \{z : z \in \mathbb{C}\ , |z| \geq a,\ |z - 1| \geq 1 - b\}$ to $\mathcal{L}(\mathcal{B}_{\theta'}, \mathcal{B}_\theta)$ such that

$$\lim_{t\to 0} \frac{1}{t^2} \sup_{z\in\mathcal{R}} \left\| (z - P(t))^{-1} - (z - P)^{-1} - t\, R'_z - \frac{t^2}{2} R''_z \right\|_{\theta',\theta} = 0.$$

Proof. (a) The proof of (a) is easy, let us give it for $n = 1$. Define the kernel

$$Lf(x) = iP(\xi f)(x).$$

Let $\varepsilon \in]0, 1]$. From the inequality $|e^{iu} - 1 - iu| \leq C_\varepsilon |u|^{1+\varepsilon}$ that is valid for all $u \in I\!R$, we get for $f \in \mathcal{B}_{\theta'}$

$$\begin{aligned} |P(t)f - Pf - tLf| &\leq P\Big(|e^{it\xi} - 1 - it\xi|\,|f|\Big) \\ &\leq C_\varepsilon |t|^{1+\varepsilon}\, \|f\|_{\theta'} P\Big(|\xi|^{1+\varepsilon} V^{\theta'}\Big) \leq D^{1+\varepsilon}\, C_\varepsilon |t|^{1+\varepsilon}\, \|f\|_{\theta'} P\Big(V^{\theta'+\alpha(1+\varepsilon)}\Big). \end{aligned}$$

With ε such that $\theta' + \alpha(1+\varepsilon) \leq \theta$, we have $PV^{\theta'+\alpha(1+\varepsilon)} \leq PV^{\theta} \leq \|P\|_{\theta} V^{\theta}$. Consequently we have $\|P(t)f - Pf - itLf\|_{\theta} \leq D^{1+\varepsilon} C_{\varepsilon}|t|^{1+\varepsilon} \|P\|_{\theta} \|f\|_{\theta'}$. This proves (a) for $n = 1$.
The proof of Assertions (b)-(c) is based on the following relation which holds for all bounded operators U and V on a Banach space such that U and $U - V$ are invertible :

$$(\mathbf{A}) \qquad (U-V)^{-1} = \sum_{k=0}^{n} (U^{-1}V)^k U^{-1} + (U^{-1}V)^{n+1}(U-V)^{-1}.$$

((A) is an easy consequence of the formula $I - W^{n+1} = (I - W) \sum_{k=0}^{n} W^k$).

To prove (b), let us apply (A) with $n = 1$, $U = z - P$, $V = P(t) - P$, thus $U - V = z - P(t)$. Using the notations

$$R(z,t) = (z - P(t))^{-1} \quad \text{and} \quad R(z) = (z - P)^{-1},$$

and those of Proposition 1 (applied to $P(t)$ acting on $\mathcal{B}_{\theta'}$), we obtain for $z \in \mathcal{R}_{\theta'}$ and $t \in I_{\theta'}$

$$R(z,t) = R(z) + R(z)(P(t) - P)R(z) + R(z)(P(t) - P)R(z)(P(t) - P)R(z,t).$$

Since by hypothesis $\theta' + \alpha < \theta$, one may choose θ_1 such that $\theta' < \theta_1 \leq \theta_1 + \alpha < \theta$. Thus assertion (a) applies to (θ', θ_1) and $n = 0$:

$$\lim_{t \to 0} \|P(t) - P\|_{\theta',\theta_1} = 0.$$

Since condition $\mathcal{U}_1(\theta_1, \theta)$ holds, Assertion (a) applied to (θ_1, θ) and $n = 1$ involves

$$P(t) - P = tL + \Upsilon(t), \quad \text{with } L, \Upsilon(t) \in \mathcal{L}(\mathcal{B}_{\theta_1}, \mathcal{B}_{\theta}) \text{ and } \lim_{t \to 0} |t|^{-1} \|\Upsilon(t)\|_{\theta_1,\theta} = 0.$$

Now let us write

$$R(z,t) = R(z) + tR'_z + \Theta_1(z,t) + \Theta_2(z,t),$$

with $R'_z = R(z)LR(z)$, and

$$\Theta_1(z,t) = R(z)\Upsilon(t)R(z), \qquad \Theta_2(z,t) = R(z)(P(t) - P)R(z)(P(t) - P)R(z,t).$$

The real numbers a, b of the statement are chosen such that the corresponding domain $\mathcal{R}$ is contained in each set $\mathcal{R}_{\theta'}, \mathcal{R}_{\theta_1}, \mathcal{R}_{\theta}$ (See Prop. 1). Besides consider the interval $I = I_{\theta'} \cap I_{\theta_1} \cap I_{\theta}$.
Since we have $L \in \mathcal{L}(\mathcal{B}_{\theta_1}, \mathcal{B}_{\theta}) \subset \mathcal{L}(\mathcal{B}_{\theta'}, \mathcal{B}_{\theta})$, and since $R(\cdot)$ is continuous from $\mathcal{R}$ to both $\mathcal{L}(\mathcal{B}_{\theta'})$ and $\mathcal{L}(\mathcal{B}_{\theta})$, the function $z \mapsto R'_z$ is continuous from $\mathcal{R}$ to $\mathcal{L}(\mathcal{B}_{\theta'}, \mathcal{B}_{\theta})$.
Using the preceding and the last assertion of Proposition 1, we obtain for $t \in I$ and $z \in \mathcal{R}$

$$|t|^{-1}\|\Theta_1(z,t)\|_{\theta',\theta} \leq |t|^{-1}\|\Theta_1(z,t)\|_{\theta_1,\theta} \leq |t|^{-1} M_{\theta} \|\Upsilon(t)\|_{\theta_1,\theta} M_{\theta_1},$$

$$|t|^{-1}\|\Theta_2(z,t)\|_{\theta',\theta} \leq M_{\theta} \left(\|L\|_{\theta_1,\theta} + |t|^{-1}\|\Upsilon(t)\|_{\theta_1,\theta} \right) M_{\theta_1} \|P(t) - P\|_{\theta',\theta_1} M_{\theta'}.$$

The last members in these inequalities do not depend on $z \in \mathcal{R}$ and converge to 0 when $t \to 0$. This proves Assertion (b) of Lemma.
For assertion (c), we can proceed in the same way by applying (A) with $n = 2$ (use the estimations of Hennion and Hervé (2004) adapted to $\mathcal{B}_{\theta}$-type spaces). □

As in the standard perturbation theory, the eigen-elements $v(t), \phi(t)$, and $N(t)$ are obtained in Keller and Liverani (1999) with the help of the projections defined by

$$\Pi_j(t) = \frac{1}{2i\pi}\int_{\Gamma_j}(z-P(t))^{-1}dz, \qquad j=0,1,$$

where the line integrals are considered on some suitable oriented circles Γ_1 and Γ_0 respectively centered at $z=1$ and $z=0$. More precisely, denoting by $\Pi_1(t)^*$ the conjugate operator of $\Pi_1(t)$, we get from Keller and Liverani (1999)

$$\phi(t) = \Pi_1(t)^*\pi, \quad v(t) = \langle \pi, \Pi_1(t)\mathbf{1}\rangle^{-1}\Pi_1(t)\mathbf{1}, \text{ and for } n\geq 1, \quad N(t)^n = \frac{1}{2i\pi}\int_{\Gamma_0} z^n\,(z-P(t))^{-1}dz.$$

From Assertions (b) and (c) of Lemma 4 and by integration, we easily deduce the following properties (See Hennion and Hervé (2004)) :

Lemma 5.

(b') Under condition $\mathcal{U}_1(\theta',\theta)$, the functions $v(\cdot)$, $\phi(\cdot)$, and $N(\cdot)$ admit a TE(1) at $t=0$ in respectively $(\mathcal{B}_{\theta'}, \|\cdot\|_\theta)$, $\mathcal{B}'_{\theta'}$ and $\mathcal{L}(\mathcal{B}_{\theta'},\mathcal{B}_\theta)$. Moreover there exist some constants $K>0$ and $\rho<1$ (which depend on (θ',θ)) such that

$$\forall n\geq 1,\ \forall t\in I, \qquad \|N(t)^n - N(0)^n\|_{\theta',\theta} \leq K\,|t|\,\rho^n.$$

(c') Under condition $\mathcal{U}_2(\theta',\theta)$, the functions $v(\cdot)$, $\phi(\cdot)$ and $N(\cdot)$ admit a TE(2) at $t=0$ in respectively $(\mathcal{B}_{\theta'}, \|\cdot\|_\theta)$, $\mathcal{B}'_{\theta'}$ and $\mathcal{L}(\mathcal{B}_{\theta'},\mathcal{B}_\theta)$. Moreover we have, for all $t\in I$ and $n\geq 1$,

$$N(t)^n = N(0)^n + t\,N_{1,n} + \frac{t^2}{2}N_{2,n} + t^2\varepsilon_n(t),$$

with, for $j=1,2$: $N_{j,n},\ \varepsilon_n(t)\in\mathcal{L}(\mathcal{B}_{\theta'},\mathcal{B}_\theta)$, $\lim_{t\to 0}\sup_{n\geq 1}\|\varepsilon_n(t)\|_{\theta',\theta}=0$, and $\sup_{n\geq 1}\|N_{j,n}\|_{\theta',\theta}<+\infty$.

III. PROOF OF LIMIT THEOREMS

Using the previous preparations, we shall provide in Section III.1 Taylor's expansions of the characteristic function of S_n. Then Theorems I-II will derive from the usual Fourier transform techniques which are similar to those employed for sums of independant identically distributed random variables. These techniques are presented in Hennion and Hervé (2001) and we shall indicate in Sections III.2-4 what parts of this work may be used to establish Theorems I-II.

The spaces $\mathcal{B}_\theta$ and the Fourier kernels $P(t)$ have been defined in Sections II.1-2. The functions $\lambda(\cdot)$, $v(\cdot)$, $\phi(\cdot)$ and $N(\cdot)$ have been introduced in Proposition 1 ; for a given $0<\theta\leq 1$, they are defined from an open interval I_θ containing $t=0$ to respectively $\mathbb{C}$, $\mathcal{B}_\theta$, $\mathcal{B}'_\theta$ and $\mathcal{L}(\mathcal{B}_\theta)$. Recall that $\|\cdot\|_{\theta',\theta}$ denotes the operator norm in $\mathcal{L}(\mathcal{B}_{\theta'},\mathcal{B}_\theta)$.
We assume that ξ is dominated by V^α, that is $\xi\in\mathcal{B}_\alpha$; the conditions imposed to α will be specified in the next statements.

III.1. EXPANSIONS OF THE CHARACTERISTIC FUNCTION OF S_n

The proposition below will be only used in the case $f=\mathbf{1}$ afterwards. By applying this proposition to $f\in\mathcal{B}_\theta$, $f\geq 0$, with suitable θ, one may generalize Theorems I-II to the sequence of r.v $(X_n, S_n)_n$ as in Hennion and Hervé (2001) (2004).

Recall that μ_0 is the initial distribution of the chain, we shall suppose that $\mu_0(V) < +\infty$.
Since $\alpha \leq \frac{1}{2}$ in the following statements, the real number $m = \pi(\xi)$ and the asymptotic variance σ^2 (See Sect. I) are defined.

Proposition 3.
(1) *If $\alpha < \frac{1}{2}$, then, for every real number θ such that $\alpha < \theta < 1 - \alpha$, there exists a function $A(\cdot)$ from I_θ to $\mathcal{L}(\mathcal{B}_\theta)$ such that we have, for $n \geq 1$, $t \in I_\theta$, and $f \in \mathcal{B}_\theta$*

$$(F') \qquad I\!\!E_{\mu_0}\left[f(X_n)e^{itS_n}\right] = \langle \mu_0, P(t)^n f\rangle = \lambda(t)^n\Big(\pi(f) + < \mu_0, A(t)f >\Big) + \langle \mu_0, N(t)^n f\rangle.$$

For all $t \in I_\theta$, we have $|\lambda(t)| \leq 1$, there exists $S \in \mathbb{C}$ such that [4]

$$\lambda(t) = 1 + imt - S\frac{t^2}{2} + o(t^2),$$

and, if $m = \pi(\xi) = 0$, we have $S = \sigma^2$.
Furthermore there exist some constants $K \geq 0$ and $\rho < 1$ which depend on θ such that we have the following properties for $n \geq 1$, $t \in I_\theta$ and $f \in \mathcal{B}_\theta$:
(i) If , either $f = 1$, or $f \in \mathcal{B}_\theta$ and $\mu_0 = \pi$, then

$$|\langle \mu_0, N(t)^n f\rangle| \leq K\,\rho^n\,|t|\,\mu_0(V)\,\|f\|_\theta$$

(ii) $\|N(t)^n\|_\theta \leq K\,\rho^n$,
(iii) $\|A(t)\|_{\theta,1} \leq K\,|t|$.

Moreover, if $m = 0$ and $\sigma^2 > 0$, then, for each real number t such that $\frac{t}{\sigma} \in I_\theta$, we have
(iv) $\quad |\lambda(\dfrac{t}{\sigma})| \leq e^{-\frac{t^2}{4}}$.

(2) *If $\alpha < \frac{1}{4}$, and if $m = 0$ and $\sigma^2 > 0$, then there exists a constant C such that we have, for each real number t satisfying $\frac{t}{\sigma\sqrt{n}} \in I_\theta$,*
(v) $\quad |\lambda(\dfrac{t}{\sigma\sqrt{n}})^n - e^{-\frac{t^2}{2}}| \leq \dfrac{C}{\sqrt{n}}|t|^3 e^{-\frac{t^2}{4}}$.

For the proof of Proposition 3, we proceed as in Hennion and Hervé (2004), we shall just recall the main arguments.

Proof of Proposition 3.
(1) Since $\alpha < \frac{1}{2}$ and $\alpha < \theta < 1 - \alpha$, one may find θ_2 such that $0 < \theta_2 \leq \theta_2 + \alpha < \theta \leq \theta + \alpha < 1$. The conditions $\mathcal{U}_1(\theta, 1)$ and $\mathcal{U}_1(\theta_2, \theta)$ of Section II.4 then hold. This yields the following properties (by Lemmas 4-5) :

(R1) *The functions $v(\cdot)$, $\phi(\cdot)$, and $N(\cdot)$ have a TE(1) at $t = 0$ in respectively $(\mathcal{B}_\theta, \|\cdot\|_1)$, $\mathcal{B}'_\theta$ and $\mathcal{L}(\mathcal{B}_\theta, \mathcal{B}_1)$, and there exist some constants $K \geq 0$ and $\rho < 1$ such that we have, for all $n \geq 1$ and $t \in I_\theta$, $\|N(t)^n - N(0)^n\|_{\theta,1} \leq K\,|t|\,\rho^n$.*

(R2) *$P(\cdot)$ has a TE(1) at $t = 0$ in $\mathcal{L}(\mathcal{B}_{\theta_2}, \mathcal{B}_\theta)$.*

[4]It can be proved that $S \in I\!\!R_+$ and $S \geq m^2$, See for instance Hennion and Hervé (2004) (Lemma 9.5).

The first equality in (F') easily follows from the Markov property, See for instance Hennion and Hervé (2001), the second one results from Proposition 1 by setting $A(t)f = \langle \phi(t), f\rangle v(t) - \langle \pi, f\rangle \mathbf{1}$. The conditions in Assertion (i) involve that $\langle \mu_0, N(0)^n f\rangle = 0$, consequently the inequality in (i) is an immediate consequence of (R1). The inequality (ii) has already been stated in Proposition 1. The inequality (iii) can be proved by observing that, since $\phi(\cdot)$ and $v(\cdot)$ have a TE(1) at $t = 0$ in respectively $\mathcal{B}'_\theta$ and $(\mathcal{B}_\theta, \|\cdot\|_1)$, there exist some constants C_1 and C_2 such that we have for $f \in \mathcal{B}_\theta$

$$\begin{aligned} \|A(t)f\|_1 &\leq |\langle \phi(t), f\rangle| \, \|v(t) - \mathbf{1}\|_1 \; + \; |\langle \phi(t) - \pi, f\rangle| \, \|\mathbf{1}\|_1 \\ &\leq C_1 \, |t| \, \|\phi(t)\|_\theta \, \|f\|_\theta \; + \; C_2 \, |t| \, \|f\|_\theta \, \|\mathbf{1}\|_1. \end{aligned}$$

It remains to prove the properties concerning $\lambda(\cdot)$. We know that $\lambda(0) = 1$ and $\lambda(t)^n = \langle \pi, P(t)^n v(t)\rangle$ (Prop. 1). From the invariance of π, it follows that $|\lambda(t)|^n \leq \langle \pi, P^n |v(t)|\rangle = \langle \pi, |v(t)|\rangle$ for all $n \geq 1$, hence $|\lambda(t)| \leq 1$. Now set

$$p(t) = \langle \phi(t), \mathbf{1}\rangle, \qquad \hat{\pi}(t) = \langle \pi, P(t)\mathbf{1}\rangle = \pi(e^{it\xi}) \qquad \text{and} \qquad u(t) = P(t)\mathbf{1} - \hat{\pi}(t)\mathbf{1},$$

notice that $u(0) = 0$ and $\langle \pi, u(t)\rangle = 0$. Besides it follows from (R2) that $P(\cdot)1$ and $\langle \pi, P(\cdot)1\rangle$ have a TE(1) at $t = 0$ in respectively $\mathcal{B}_\theta$ and $\mathbb{C}$. Thus $u(\cdot)$ has a TE(1) at $t = 0$ in $\mathcal{B}_\theta$.
In order to prove the stated expansion for $\lambda(\cdot)$, we shall use the following properties (R3)-(R4) whose proof is easy (use Prop 1. for (R3) and Lebesgue Theorem for (R4)) :

(R3) $\lambda(t) = \dfrac{1}{p(t)} \langle \phi(t) - \pi, u(t)\rangle + \hat{\pi}(t)$.

(R4) *If $n\alpha \leq 1$, then $\hat{\pi}(\cdot)$ is of class $\mathcal{C}^n$, and $\hat{\pi}^{(k)}(0) = i^k \pi(\xi^k)$ for $k = 1, \ldots, n$.*

Lemma 6. *Let $S = \pi(\xi^2) - 2\langle \phi'(0), u'(0)\rangle$. Then $\lambda(t) = 1 + imt - S\frac{t^2}{2} + o(t^2)$. Moreover, if $m = \pi(\xi) = 0$, then $S = \sigma^2$.*

Proof. By (R4), from the fact that $\alpha < \frac{1}{2}$, we have $\hat{\pi}(t) = 1 + imt - \pi(\xi^2)\frac{t^2}{2} + o(t^2)$. We have $\phi(t) - \pi = \phi(t) - \phi(0) = t\phi'(0) + o(t)$ in $\mathcal{B}'_\theta$, $u(t) = tu'(0) + o(t)$ in $\mathcal{B}_\theta$, and $p(t) = 1 + O(t)$ in $\mathbb{C}$. Setting $c = 2\langle \phi'(0), u'(0)\rangle$, we get

$$\frac{1}{p(t)} \langle \phi(t) - \pi, u(t)\rangle = \Big(1 + O(t)\Big)\Big(c\frac{t^2}{2} + o(t^2)\Big) = c\frac{t^2}{2} + o(t^2).$$

Combining in (R3) the previous expansions, we obtain $\lambda(t) = 1 + imt - S\,\frac{t^2}{2} + o(t^2)$.
The equality $S = \sigma^2$, under the condition $m = 0$, is a well-known fact when the standard perturbation theory of operators is applied, See for instance Ney and Nummelin (1987), Kontoyiannis and Meyn (2003). The arguments used in these papers easily extend to the present context. □

To prove the property (iv) of Proposition 3, observe that, if $m = 0$ and $\sigma^2 > 0$, then we have, for small $|t|$, $|\lambda(\frac{t}{\sigma})| \leq 1 - \frac{t^2}{2} + \frac{t^2}{4} \leq e^{-\frac{t^2}{4}}$.
To prove Assertion (2) of Proposition 3, it suffices to establish that $\lambda(t) = 1 - \sigma^2\frac{t^2}{2} + O(t^3)$, See for instance Hennion and Hervé (2001), Section VI.2. For that, notice that, under the condition $\alpha < \frac{1}{4}$, there exist θ_4 and θ_2 such that $0 < \theta_4 \leq \theta_4 + 2\alpha < \theta_2 \leq \theta_2 + 2\alpha < 1$. So the conditions $\mathcal{U}_2(\theta_2, 1)$ and $\mathcal{U}_2(\theta_4, \theta_2)$ hold, and we get by Lemmas 4-5 : $\phi(t) - \pi = t\phi'(0) + t^2\phi_2 + o(t^2)$ in $\mathcal{B}'_{\theta_2}$ and $u(t) = tu'(0) + t^2 u_2 + o(t^2)$ in $\mathcal{B}_{\theta_2}$. Hence $\frac{1}{p(t)}\langle \phi(t) - \pi, u(t)\rangle = (1 + O(t))(c\frac{t^2}{2} + O(t^3)) = c\frac{t^2}{2} + O(t^3)$. Since $\hat{\pi}(\cdot)$ is of class $\mathcal{C}^4$ by (R4), the stated expansion for $\lambda(t)$ follows from (R3). □

III.2 PROOF OF THEOREM I.

To make easier the link with the proofs given in Hennion and Hervé (2001), let us specify that Proposition 3 exactly corresponds to Proposition VI.2 in the previously cited work. Let us just notice that $\lambda = 1$ is here the unique peripheral eigenvalue of P acting on $\mathcal{B}_\theta$; the perturbed eigenvalue $\lambda(t)$ is denoted by $\lambda_1(t)$ in Hennion and Hervé (2001).
Besides observe that, for all $\theta \in]0,1]$, we have $\mathbf{1} \in \mathcal{B}_\theta$, and $\mu_0 \in \mathcal{B}'_\theta$ since $\mu_0(V) < +\infty$.

Following the proof of Section VI.3 in Hennion and Hervé (2001), Theorem I is a consequence of Proposition 3 applied with $f = \mathbf{1}$.

III.3 PROOF OF THEOREM II.

In order to establish Theorems II, we shall investigate the link between the condition (N-A)$_\theta$ of Section I and the spectral properties of $P(t)$.

Lemma 7. *Let $0 < \theta \leq 1$. Assume that the condition (N-A)$_\theta$ holds. Then, for all compact subset K of $\mathbb{R}^*$, there exist $c_{K,\theta} \geq 0$ and $\rho_{K,\theta} < 1$ such that we have, for all $n \geq 1$,*

$$\text{(N-A)}'_\theta \qquad \sup_{t\in K} \|P(t)^n\|_\theta \leq c_{K,\theta}\, \rho^n_{K,\theta}.$$

In particular, if $\mu_0(V) < +\infty$, then we have $\sup_{t\in K} |\langle \mu_0, P(t)^n \mathbf{1}\rangle| \leq c_{K,\theta}\, \mu_0(V)\, \|\mathbf{1}\|_\theta\, \rho^n_{K,\theta}$.

Theorem II then results from Proposition 3 (applied with $f = 1$) and from Lemma 7 which enable to employ the Fourier transform techniques presented in Section VI.4 in Hennion and Hervé (2001). Notice that, in Theorems II, the hypothesis (N-A)$_\theta$ may be replaced with (N-A)$'_\theta$.

Proof of lemma 7. Let $r(P(t))$ be the spectral radius of $P(t)$ acting on $\mathcal{B}_\theta$.

1. For all $t \in \mathbb{R}^*$, we have $r(P(t)) < 1$.
From the inequality (KL1) of Proposition 2, we obtain that the sequence $(P(t)^n)_n$ is bounded in $\mathcal{L}(\mathcal{B}_\theta)$. Thus $r(P(t)) \leq 1$. Suppose now that $r(P(t)) = 1$: then $P(t)$ is quasi-compact by (KL2), thus there exists an eigenfunction $w \in \mathcal{B}_\theta$ corresponding to an eigenvalue $\lambda \in \mathbb{C}$ of modulus one, but this is impossible under the condition (N-A)$_\theta$ (See for instance Hennion and Hervé (2001) Prop. V.2).

2. Let K be a compact subset of $\mathbb{R}^*$. We have $r_K = \sup\{r(P(t)),\ t \in K\} < 1$.
Suppose that $r_K = 1$. Then there exists a sequence $(\tau_k)_k$ in K such that $\lim_k r(P(\tau_k)) = 1$. For $k \geq 1$, denote by λ_k a spectral value of $P(\tau_k)$ such that $|\lambda_k| = r(P(\tau_k))$. By compactness, one may assume that the sequences $(\tau_k)_k$ and $(\lambda_k)_k$ converge. Let $t_0 = \lim_k \tau_k$ and $\lambda = \lim_k \lambda_k$; observe that $t_0 \in K$, thus $t_0 \neq 0$, and $|\lambda| = 1$. Besides, when $t \to t_0$, the family $\{P(t),\ t \in \mathbb{R}\}$ equally verifies the conditions of Proposition 2 (that is, (KL1) (KL2) (KL3) hold in the same way, and it is easily seen that (KL4) remains valid when $t \to t_0$ and P is replaced with $P(t_0)$). Then it follows from Keller and Liverani (1999) (p. 145) that λ is a spectral value of $P(t_0)$. But, since $t_0 \neq 0$ and $|\lambda| = 1$, this contredicts assertion *1.*

3. There exist $c_K \geq 0$ and $\rho_K < 1$ such that we have, for all $n \geq 1$, $\sup_{t\in K} \|P(t)^n\|_\theta \leq c_K\, \rho^n_K$.

Let ρ_K be such that $r_K < \rho_K < 1$, and let Γ be the oriented circle $\{|z| = \rho_K\}$ in $\mathbb{C}$. We have

$$P(t)^n = \frac{1}{2i\pi} \int_\Gamma z^n (z - P(t))^{-1} dz.$$

Let $t_0 \in K$. The results of Keller and Liverani (1999), applied as above with $t \to t_0$, ensures that there exists an open interval I_{t_0} containing t_0 such that $\sup\{\|(z - P(t))^{-1}\|_\theta,\ t \in I_{t_0},\ |z| = \rho_K\} < +\infty$. From compactness it follows that $\sup\{\|(z - P(t))^{-1}\|_\theta,\ t \in K,\ |z| = \rho_K\} < +\infty$. The property *3* then derives from the above integral formula. □

REFERENCES

Bolthausen, E. (1982). The Berry-Esseen theorem for strongly mixing Harris recurrent Markov chains. *Z. Wahrscheinlichkeitstheorie verw. Gebiet* **60**, *283-289.*

Chen, X. (1999). Limit Theorems for Functionals of Ergodic Markov Chains with General State Space. *Memoirs of A.M.S. Vol 139, No. 664.*

Fuh, C.D.(1999). Paley-type inequalities related to the central limit theorem for markov chains. *Sankhya : The Indian Journal of Statistics,* **61**, *Series A, Pt. 1, 89-100.*

Guivarc'h, Y. and Hardy, J. (1988). Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov. *Ann. Inst. Henri Poincaré, 24, 1, pp. 73-98.*

Hennion, H. Quasi-compactness and absolutely continuous kernels.
I.R.M.A.R, Université de Rennes I.

Hennion, H. and Hervé, L. (2001) Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. *Lecture Notes in Mathematics No 1766, Springer.*

Hennion, H. and Hervé, L. (2004). Central Limit theorems for iterated random lipschitz mappings. *Annals of Probability,* **32**, *No. 3A, 1934-1984.*

Hervé, L. (2005) Théorème local pour chaînes de Markov de probabilité de transition quasi-compacte. Applications aux chaînes V-géométriquement ergodiques et aux modèles itératifs.
Ann. I. H. Poincaré - PR 41, 179-196.

Hervé, L. (2006) Vitesse de convergence dans le théorème limite central pour chaînes de Markov de probabilité de transition quasi-compacte. Preprint, IRMAR.

Keller G. and Liverani C. (1999). Stability of the Spectrum for Transfer Operators. *Ann. Scuola Norm. Sup. Pisa. CI. Sci. (4) Vol. XXVIII, 141-152.*

Kontoyiannis, I. and Meyn, S.P. (2003). Spectral theory and limit theorems for geometrically ergodic Markov processses. *Annals of Applied Probability,* **13**, *304-362.*

Kontoyiannis, I. and Meyn, S.P. (2005). Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electronic Journal of Probability, Vol. 10.*

Meyn, S.P. and Tweedie, R.L. (1993). Markov chains and stochastic stability. *Springer Verlag, New York, Heidelberg, Berlin.*

Meyn, S.P. and Tweedie, R.L. (1994). Computable bounds for convergence rates of Markov chains. *Annals of Applied Probability,* **4**, *981–1011.*

Milhaud, X. and Raugi, A. (1989). Etude de l'estimateur du maximum de vraisemblance dans le cas d'un processus auto-régressif : convergence, normalité asymptotique, vitesse de convergence. *Ann. Inst. H. Poincaré,* **25**, *4, 383-428.*

Nagaev, S.V. (1957). Some limit theorems for stationary Markov chains. *Theory of probability and its applications,* **11**, *4, 378-406.*

Ney P. and Nummelin E. (1987). Markov additive processes. I. Eigenvalue properties and limit theorems. *Ann. Probab.,* **15**, *No. 2, 561-592.*

Steinsaltz, D. (2001). Convergence of moments in a Markov-chain central limit theorem. *Indagationes Mathematicae,* **12**, *533-555.*